Topics in complex analysis   Lecture 11   (Dec 2, 2024)

Definition 8.1. Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$. Then f is called complex-differentiable in $a \in U$ if there exists a $\underline{\mathbb{C}\text{-linear}}$ map $Df(a) : \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{\|h\|} = 0.$$

f is called holomorphic on U if f is complex differentiable in every point $a \in U$. A function $f : U \to \mathbb{C}^m$ is called holomorphic if each component is holomorphic.

<u>Prop</u> (Properties of holomorphic fcn & their differentials)

Let $f : U \longrightarrow \mathbb{C}$ holomorphic, $U \subseteq \mathbb{C}^n$ open.

(i) uniqueness. $\forall a \in U$, $Df(a)$ is unique.

(ii) Linearity. If $g : U \longrightarrow \mathbb{C}$ is holo and $\lambda \in \mathbb{C}$, then

$$D(f+\lambda g)(a) = Df(a) + \lambda Dg(a) \quad \forall a \in U.$$

(iii) chain rule. If $\varphi : V \longrightarrow \mathbb{C}$, $\underline{V}$ open set containing $f(U)$, holo, then

can be chosen $f(U)$ if f nonconst

$$D(\varphi \circ f)(a) = \varphi' \circ f(a)\, Df(a).$$

Accordingly if $\psi : W \longrightarrow U$, $W \subseteq \mathbb{C}^m$ open, holo

$$D(f \circ \psi)(b) = \underbrace{Df(\psi(b))\, D\psi(b)}_{\text{matrix multipl.}}$$

(iv) Leibniz rule. If $g : U \longrightarrow \mathbb{C}$ holo. then

$$D(fg)(a) = f(a)\,Dg(a) + g(a)\,Df(a) \quad \forall a \in U.$$

Theorem 8.5 (Cauchy's integral formula for polydiscs). *Let $U \subset \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let $a \in U$ and $r \in (0, +\infty)^n$ be such that $\overline{D_r^n(a)} \subset U$. Then for all $z \in D_r^n(a)$ it holds that*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_n - a_n| = r_n} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta)}{\prod_{i=1}^n (\zeta_i - z_i)} \, \mathrm{d}\zeta.$$

Proof: Induction over the dim n.

For $n=1$ this is already known (Thm 0.2).

Next we assume $n \geq 2$. Given $z \in D_r^n(a)$ by def of $D_r^n(a)$ we have $\underline{z_n \in B_{r_n}(a_n)}$ and $\underline{(z_1, \ldots, z_{n-1}) \in D^{n-1}_{(r_1, \ldots, r_{n-1})}(a_1, \ldots, a_{n-1})}$

will be used later to apply 1-dim. Cauchy integral formula

will be used to apply the induction hypothesis.

Note the closure of $D^{n-1}_{(r_1, \ldots, r_{n-1})}(a_1, \ldots, a_{n-1})$ is contained in $U_{n-1} = \{ z \in \mathbb{C}^{n-1} : (z, z_n) \in U \}$. Moreover the function

$$U_{n-1} \ni (z_1, \ldots, z_{n-1}) \longmapsto f(z_1, \ldots, z_{n-1}, \underbrace{z_n}_{\text{fixed!}}) \text{ is holo } \forall z_n \in B_{r_n}(a_n)$$

By induction hypothesis

$$f(z_1, \ldots, z_{n-1}, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{|\zeta_{n-1} - a_{n-1}| = r_{n-1}} \cdots \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, \ldots, \zeta_{n-1}, z_n)}{\prod_{i=1}^{n-1} (\zeta_i - z_i)} \, d\zeta_1 \cdots d\zeta_{n-1}$$

This nominator is again holo. in z_n for fixed(!) $\zeta_1, \ldots, \zeta_{n-1}$. Applying the 1-dim Cauchy integral formula, the claim follows. $\square$

Rem (multiindex notation) For $\alpha \in \mathbb{N}_0^n$, $z \in \mathbb{C}^n$ we write

- $\alpha! := \alpha_1! \cdots \alpha_n!$
- $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$
- $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$

Theorem 8.9 (Special case of Hartogs' extension theorem). *Let $D \subset \mathbb{C}^{n-1}$ be a domain and $A(r,R) := \{z \in \mathbb{C} : r < |z| < R\}$ with $0 \le r < R \le +\infty$. Let $f : D \times A(r,R) \to \mathbb{C}$ be holomorphic. Assume that there exists $a \in D$ and $\varepsilon > 0$ such that f can be extended holomorphically to $B_\varepsilon(a) \times B_R(0)$. Then f can be extended holomorphically to $D \times B_R(0)$.*

Proof We write (z', z_n) for a generic pt in $D \times \mathbb{C}$. Define

DCT

$$f_\rho(z', z_n) := \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(z', \zeta)}{\zeta - z_n} \, d\zeta$$

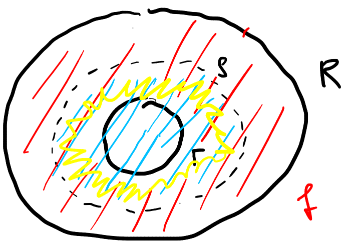
DCT plus bddness away from zero

where $r < \rho < R$.

This fct is continuous on $D \times B_\rho(0)$ and separately holomorphic in each variable (DCT). By Hartogs thm (black box, Thm 8.13) it is holo on $D \times B_\rho(0)$.

By our hypothesis $\exists \varepsilon > 0$ st f can be extended to $B_\varepsilon(a) \times B_R(0)$. Applying the 1-dim Cauchy integral formula yields $f = f_\rho$ on $B_\varepsilon(a) \times B_\rho(0)$. (keeping the argument in $B_\varepsilon(a)$ fixed). But now $D \times A(r,\rho)$ is a domain in $\mathbb{C}^n$ st the identity thm yields $f = f_\rho$ on $D \times A(r,\rho)$. Then the function

$$F(z', z_n) = \begin{cases} f(z', z_n) & \text{if } (z', z_n) \in D \times A(r,R) \\ f_\rho(z', z_n) & \text{if } (z', z_n) \in D \times B_\rho(0) \end{cases}$$



extends f, is well-def, and holomorphic. $\square$